A Method of Solution for Certain Problems of Transient Heat Conduction

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This paper develops a numerical treatment of classical boundary value problems for arbitrarily shaped plane heat conducting solids obeying Fourier's law. An exact integral formula defined on the boundary of an arbitrary body is obtained from a fundamental singular solution to the governing differential equation. This integral formula is shown to be a means of numerically determining boundary data, complementary to given *data*, such that the Laplace transformed temperature field may subsequently be generated by a Green's type integral identity. The final step, numerical transform inversion, completes the solution for a given problem. All operations are ideally suited for modern digital computation. Three illustrative problems are considered. Steady-state problems, for which the Laplace transform is unnecessary, form a relatively simple special case.

Introduction

A FORMULATION of the various transient boundary value problems associated with isotropic solids obeying Fourier's law of heat conduction is developed. An exact integral formula is derived relating boundary heat flux *and* boundary temperature, in the Laplace transform space, that corresponds to the same admissible transformed temperature field throughout the body. Part of the boundary data in the formula is known from the description of a well posed boundary value problem. As is shown, the remaining part of the boundary data is obtainable numerically from the formula itself regarded as a singular integral equation. Once both transformed temperature *and* heat flux are known everywhere on the boundary, the transformed temperature throughout the body is obtainable by means of a Green's type integral identity. This identity yields the field directly in terms of the mentioned boundary data. The final step, transform inversion, although done approximately also, is accomplished by a technique particularly well suited to the class of problems under investigation.

The main feature of the solution procedure suggested is its generality. It is applicable to solids occupying domains of rather arbitrary shape and connectivity. Boundary data may be prescription of temperature, or heat flux, or parts of each corresponding to a mixed type problem. Also, a linear combination of temperature and flux may be given corresponding to the so-called convection boundary condition. The *same* boundary formula described previously is applicable in every case. Approximations in the transform space are made only on the boundary, in contrast to finite difference procedures, and the approximations made are conceptually simple, natural to make, and give rise, as is shown, to very accurate data for a relatively crude boundary approximation pattern. Problems posed for composite bodies, i.e., two or more heat conducting solids bonded together, are particularly amenable to the present treatment. One computer program is employed which utilizes only data describing the domain geometry, boundary temperature or flux, material properties, and a sequence of values of the transform parameter necessary for the inversion scheme. Output is the transformed temperature at any desired field point. A second program inverts the transformed temperature numerically to yield the final transient solution.

Although details of the mathematical development are given only for two spatial dimensions, the basic idea behind the present method is not limited to plane problems. Three illustrative problems are considered and data are compared with known analytical solutions. Finally, steady-state problems, while not necessarily trivial, are quite easily handled by a considerably simplified version of the transient formulation. The manner of treating such problems is indicated.

Formulation

Let a homogeneous isotropic heat conducting solid occupy a region D of the plane; the boundary C of D is considered, at present, to have a continuously turning tangent. Using Fourier's law of heat conduction, i.e.,

$$q(p,t) = -k(\partial T/\partial n)(p,t) \tag{1}$$

an energy balance throughout D (in the absence of internal heat sources) requires (c.f. Ref. 1, p. 140)

$$k\nabla^2 T(p,t) = \rho c(\partial T/\partial t)(p,t) \tag{2}$$

where q is heat flux at point p at time t, and T is absolute temperature. The constants k, ρ, and c are, respectively, the conductivity, mass density, and specific heat. A linear boundary condition of the form

$$M(\partial T/\partial n)(Q,t) + NT(Q,t) = G(Q,t), Q \text{ on } C \tag{3}$$

includes the most commonly specified boundary conditions for physical problems by a proper choice of constants M and N and function G. For example, $M = -k$, $N = 0$, and $G = q$, where q is known on C, yields the heat input boundary condition. $N = 1$, $M = 0$, and $G = f$, where f is known on C yields the boundary temperature condition. Also included is the convection condition for a solid immersed in a medium of known temperature T_0, i.e.,

$$k(\partial T/\partial n)(Q,t) = h[T_0 - T(Q,t)] \tag{4}$$

where h is a known surface heat transfer coefficient. Finally, mixed boundary data over C are possible, i.e., one type of boundary condition may be specified over a part C_1 of C with another type specified over the remaining part C_2. Uniqueness of solution of the various boundary value problems under relatively broad smoothness assumptions has been established.[2]

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Next, denote the Laplace transform of a function $g(p,t)$, when it exists,† by

$$g^*(p,s) \equiv \mathfrak{L}_s\{g(p,t)\} = \int_0^\infty g(p,t)e^{-st}\,dt \quad (5)$$

and limit attention (for subsequent numerical purposes) to real and positive s. Assuming that all pertinent functions possess Laplace transforms, Eqs. (2) and (3) in the transform space for zero $T(p,0)$ (et seq.) become, respectively,

$$\kappa\nabla^2 T^*(p,s) - sT^*(p,s) = 0 \quad (6)$$

$$M(\partial T^*/\partial n)(Q,s) + NT^*(Q,s) = G^*(Q,s) \quad (7)$$

The function

$$U^*(p,p',s) = (1/2\pi\kappa)K_0[(s/\kappa)^{1/2}r],\ \kappa = k/\rho c \quad (8)$$

in which K_0 is a modified Bessel function of the second kind, zero order, and r is the distance between arbitrary points p and p' in D, identically satisfies Eq. (6) (for $r \neq 0$) as direct substitution will verify. It is easily shown that

$$U^* = \mathfrak{L}_s\{U\}$$

where

$$U(p,p',t) = (1/4\pi\kappa t)e^{-r^2/4\kappa t} \quad (9)$$

is the temperature field due to a concentrated heat source of unit strength at p (Ref. 1, p. 166). Now if Green's reciprocal identity (c.f. Ref. 4, pp. 215–219) is applied to U^* and an arbitrary sufficiently smooth solution T^* to Eq. (6), the result is the identity

$$T^*(p,s) = \kappa \int_C [(\partial T^*/\partial n)(Q,s)U^*(p,Q,s) - T^*(Q,s)(\partial U^*/\partial n)(p,Q,s)]dS \quad (10)$$

where explicit point dependence of the function is indicated. The integral is taken keeping D on the left and n is the outward normal.

Identity Eq. (10) is of the classical Green's (third) type which expresses the basic dependent variable, here transformed temperature, in terms of its boundary values *and* boundary values of its normal derivative. Clearly, however, since *both* boundary functions are not simultaneously obtainable on C from a well posed prescription of boundary data, integrals of the type [Eq. (10)] do not explicitly yield the dependent variable in D. A formal analytical development at this point leading to an explicit solution for T^* would be to add a domain dependent function V^* to U^* with the property that dependence in Eq. (10) on whatever portion of T^* and $\partial T^*/\partial n$ is unknown on C is eliminated. An inverse Laplace transform would formally complete the solution. However, the difficulties in finding V^* (or its inverse, V) for any but the simplest domains are well known (c.f., comments in Ref. 1, p. 183). Indeed, the level of difficulty in obtaining analytical solutions to heat conduction problems by any means are well-appreciated such that approximate techniques must often be used for a specific problem. However, rather than abandon the inherent generality of Eq. (10), we seek instead to base a numerical solution upon it and to exploit that generality, via the modern computer.

Toward this end let P be an arbitrary point on C and assume $T^*(Q,s)$ and $(\partial T^*/\partial n)(Q,s)$ are differentiable on C. The order of the singularities of U^* and $\partial U^*/\partial n$ are, respectively, such that the integrals in Eq. (10) have properties identical to the simple and double layer potentials of potential theory (c.f. Ref. 4,5). Thus in the limit as $p \to P$, Eq. (10) becomes

$$T^*(P,s) + 2\kappa \int_C [T^*(Q,s)\ \partial U^*/\partial n\ (P,Q,s) - \partial T^*/\partial n\ (Q,s)U^*(P,Q,s)]dS = 0 \quad (11)$$

† For precise statements concerning existence and properties of Laplace transforms see, for example, Widder.[3]

in which improper integrals (i.e., when Q coincides with P) are understood in the sense of their limiting values (c.f. Ref. 5, p. 26). Equations (11) and (7) are now regarded as a pair of equations, defined on C, from which T^* *and* $\partial T^*/\partial n$ may be completely determined on C. If this is done, Eq. (10) yields, in fact, the solution $T^*(P,s)$ in D.

Specifically, substituting prescribed boundary data from Eq. (7) into Eq. (11), the latter becomes a singular integral equation in those parts of T^* and/or $\partial T^*/\partial n$ not prescribed. As such Eq. (11) represents a departure from the classical treatment of elliptic boundary value problems via integral equations (c.f. Ref. 4,5). Normally, simple or double layer potentials dependent on relatively obscure surface density functions are assumed at the outset where the choice is governed by the type of boundary value problem. Unknowns in the resultant integral equations are the mentioned surface densities. On the other hand, Eq. (11) provides a "compatibility" relation between the complementary functions T^* and $\partial T^*/\partial n$ for the purpose just described and thus plays a role for heat conduction problems which others of its type (that have only recently appeared, e.g., Ref. 6–8) play for potential theory and elasticity theory. The role is, of course, to provide with a single functional relation [here Eq. (11)] a mechanism, in principle, for solution of problems involving arbitrary shape and type of boundary conditions.

Generally, solution of Eq. (11) for a given problem is out of the question analytically. We should remark, however, that under the assumptions made, Eq. (11) is expected to have a unique solution for the required unknowns. Proof of this is possible by direct appeal to the theorems of Fredholm.[5] Also, the uniqueness discussion[8] is applicable since Eq. (11) is mathematically a special case of Eq. (5.5).[8]

It may also be remarked that if the region D occupied by heat conducting material is multiply connected, all of the foregoing arguments and statements may be shown to be valid. Further, a composite solid is formed by allowing the voids in the multiply connected region to be filled with material with different properties k, ρ, and c. Equations identical to (10) and (11) then exist for each separate body. Boundary conditions are then that the transformed temperature and also the transformed heat flux are continuous across the boundaries.

Finally, if the boundaries described previously fail to have a unique tangent at a number of isolated points, i.e., corners at a number of points on C, it may be shown (c.f. Ref. 5 Appendix 2) that a modification of the factor 2κ in Eq. (11) is required if P occupies a corner point. The modification is related to the angle of the corner. However, although a problem is subsequently posed for a square domain D, we never allow P to occupy any of the four corner positions such that Eq. (11) may be used without modification.

Numerical Procedures in the Transform Space

The method of approximate solution for the present class of boundary value problems is a) to replace Eq. (11) by a system of linear algebraic equations, b) to solve these equations for piecewise constant values of unknown parts of the functions T^* and $\partial T^*/\partial n$ on C, c) to perform the integrations in Eq. (10) to obtain T^* at desired p, and d) to invert T^* at p numerically for a complete approximate solution in the time domain.

Consider, first, steps (a–c). An approximate form of Eq. (11), i.e.,

$$T^*(P_\eta,s) + 2\kappa \sum_{\xi=1}^{N} \left\{ T^*(Q_\xi,s) \int_\xi \frac{\partial U^*}{\partial n}(P_\eta,Q,s)\,dS - \frac{\partial T^*}{\partial n}(Q_\xi,s) \int_\xi U^*(P_\eta,Q,s)dS \right\} = 0,\ (\eta = 1,2,\ldots N) \quad (12)$$

is obtained by assuming that T^* and $\partial T^*/\partial n$ have piecewise constant values, denoted by $T^*(Q_\xi,s)$ and $(\partial T^*/\partial n)(Q_\xi,s)$ over each of N selected intervals of C. Discrete point P_η are chosen successively within each of the intervals. Only integrals of known functions over the intervals occur such that Eq. (12) are a system of N linear algebraic equations in the variables $T^*(Q_\xi,s)$ and $(\partial T^*/\partial n)(Q_\xi,s)$. Therefore, once C is described and a value of s chosen, the integrals

$$A_{\eta\xi} \equiv \int_\xi \partial U^*/\partial n\ (P_\eta,Q.s)dS, \quad B_{\eta\xi} \equiv \int_\xi U^*(P_\eta,Q,s)d \tag{13}$$

may be evaluated (as will be discussed subsequently) by a digital computer such that $A_{\eta\xi}$, $B_{\eta\xi}$ are regarded as known coefficients of the mentioned variables. System (12) may therefore be written in matrix form

$$(\mathbf{1} + 2\kappa\mathbf{A})\mathbf{T}^* - \mathbf{B}\mathbf{T}_n^* = 0 \tag{14}$$

in which $\mathbf{T}^*$ and $\mathbf{T}_n^*$ are column matrices with elements $T^*(Q_\xi,s)$ and $(\partial T^*/\partial n)(Q_\xi,s)$, respectively; $\mathbf{A}$ and $\mathbf{B}$ are square $(N \times N)$ matrices with elements (13); and $\mathbf{1}$ is the identity matrix. If the constants M and N and the function G^* of the generalized boundary condition Eq. (7) are specified at each interval of C, N of the $2N$ items of discrete transformed boundary data $\mathbf{T}^*$ and $\mathbf{T}_n^*$ may be expressed in terms of the other N items. System (14) then becomes N equations in N unknowns. These equations are solved for the selected value of s such that both $\mathbf{T}^*$ and $\mathbf{T}_n^*$ for that value of s are now known.

The inversion procedure to be described requires a sequence of numerical values of a transformed function. Thus, Eq. (14) is actually formed and solved for a sequence s_ρ $(\rho = 1,2,\ldots M)$ of real, positive values of the transform parameter. Therefore prior to step c), Eq. (14) is solved M times such that a sequence of values of transformed boundary data, i.e., $\mathbf{T}_\rho^*$ and $\mathbf{T}_{n\rho}^*$ is subsequently known. Thus, with $\mathbf{T}_\rho^* \equiv T^*(Q_\xi,s_\rho)$ and $\mathbf{T}_{n\rho}^* \equiv (\partial T^*/\partial n)(Q_\xi,s_\rho)$ known on C the interior transformed temperature sequence is now obtained by means of Eq. (10) according to

$$T^*(p,s_\rho) = \kappa \sum_{\xi=1}^{N} \left\{ \frac{\partial T^*}{\partial n}(Q_\xi,s_\rho) \int_\xi U^*(p,Q,s_\rho)dS - T^*(Q_\xi,s_\rho) \int_\xi \frac{\partial U^*}{\partial n}(p,Q,s_\rho)dS \right\} \tag{15}$$

Note, no additional approximation is involved in Eq. (15).

The key issue in the aforementioned discussion of approximations is the actual evaluation of the integrals (13) and (15). From Eq. (8), it is apparent that integrals (13) may be written more explicitly in the form

$$A_{\eta\xi} = -1/2\pi\kappa \int_\xi (s/\kappa)^{1/2}\, \partial r/\partial n\ K_1[(s/\kappa)^{1/2}r]dS,$$
$$B_{\eta\xi} = 1/2\pi\kappa \int_\xi K_0[(s/\kappa)^{1/2}r]dS \tag{16}$$

where r is the distance between P_η and Q. The points P_η and Q_ξ are chosen at the middle of their associated intervals. Clearly, P_η will coincide with Q_ξ, i.e., intervals $\xi = \eta$, once in each equation of the system (14). The integrals (16) then become improper and special attention is required in their evaluation.

First note that the easily established identity

$$\partial r/\partial n \equiv r\ d\theta/dS \tag{17}$$

where θ is the angle of r with the horizontal at P_η, casts $A_{\eta\xi}$ in the form

$$A_{\eta\xi} = -1/2\pi\kappa \int_\xi [(s/\kappa)^{1/2}r]\ K_1[(s/\kappa)^{1/2}r]d\theta \tag{18}$$

For $\xi \neq \eta$, both $A_{\eta\xi}$, given by Eq. (18), and $B_{\eta\xi}$ are obtained using a 3-point Simpson approximation. The true arc length ΔS between points is used for $B_{\eta\xi}$ rather than merely the chord length while the angle change $\Delta\theta$ between points, with P_η as vertex, is used for $A_{\eta\xi}$. For $\xi = \eta$, the assumption $dS \doteq dr$ is made and since the singularity of K_0 at $r = 0$ is logarithmic, the integral, although improper,

$$B_{\eta\eta} = 2\int_0^{\Delta r} K_0\left[\left(\frac{s}{\kappa}\right)^{1/2} r\right] dr \tag{19}$$

exists and is easily integrated numerically (c.f. Ref. 9, p. 379 and Ref. 10, Table 4.6). Further, the form of $A_{\eta\xi}$ from Eq. (18) illustrates that since the singularity of $K_1(z)$ is of the order $1/z$, the integral $A_{\eta\eta}$ also exists and can be evaluated approximately like $B_{\eta\eta}$. Finally, the integrals in Eq. (15), since they involve no singularities, are all evaluated using a Simpson approximation as for the corresponding $(\xi \neq \eta)$ boundary integrals.

Required input to the computer for all approximations described thus far are cartesian coordinates of Q_ξ, $Q_{\xi-1/2}$, $Q_{\xi+1/2}$ and selected p; values of ΔS, N, and κ; the elements in the sequence s_ρ; and, finally, known elements of discretized boundary data. The manner of choosing s_ρ will be described in a following section.

Steady-State Problems

If the temperature in a heat conducting solid, as considered previously, is independent of time t Eq. (2) reduces to Laplace's equation, and the boundary condition (3) remains the same except that the functions are now time independent. Using the function $W = \log r(p,p')$ as a replacement for $U^*(p,p',s)$ and $T(p)$ as a replacement for $T^*(p,s)$, a development, parallel in every respect, may now be carried out through Eq. (15). Details, of course, are considerably simpler and no transform is involved. Although the context of the present discussion is steady state heat conduction, the outlined reduction leads to a formulation which is essentially the same‡ as given by Jaswon and Ponter[6] for the St. Venant torsion problem. Examination of the numerical data present in Ref. 6 will give an indication of the good accuracy one may expect in the solution of steady state problems by the present techniques.

Transform Inversion

The problem of inverting a general transform function, known only numerically, is treated in the survey article by Cost.[11] He discusses, at least briefly, most of the important methods of approximate Laplace transform inversion including those proposed by Widder,[8] Alfrey,[12] Papoulis,[13] and Schapery.[14] Bellman[15] also provides a lucid account of the pertinent aspects of transform inversion in general. From the cited work, it is quite clear that no existing method of approximate inversion can be expected to provide acceptable results for completely arbitrary transform functions. However, under certain frequently reasonable (and fortunately not too severe) restrictions on the character of the transform functions, or perhaps on the function of time sought, excellent results are obtainable by the cited schemes as well as others.[11,15]

For the example heat conduction problems to follow, we adapt the "collocation" inversion method of Schapery[14] proposed for viscoelastic stress analysis and seek solutions for temperature pointwise in the form

$$g(t) = A + Bt + \theta(t) \tag{20}$$

in which A and B are constants and $\theta(t)$ is called the "transient" part of $g(t)$. We further assume $\theta(t)$ to be approximately representable by the finite Dirichlet Series

$$\theta(t) \doteq \sum_{\rho=1}^{m} a_\rho e^{-b\rho t} \tag{21}$$

‡ Some suggestions for the numerical evaluation of the elements of the matrices **A** and **B** given above differ with Ref. 6. For example, explicit introduction and use of the variable θ has allowed a numerically more convenient, efficient and accurate evaluation of the **A** elements than is proposed in Ref. 6.

where a_ρ and b_ρ are constants. A necessary and sufficient condition[14,16] for a function $\theta(t)$ to be expandable in a convergent Dirichlet series is that the integral of $\theta^2(t)$ over all positive t exists. It may be argued, (c.f. Ref. 1, Chap. 7) that heat conduction problems involving boundary functions G which are constant functions of time have solutions with transient parts that meet this requirement. For more general $G(Q,t)$, we may obtain the solution from that for a constant time input by means of the Duhamel superposition integral (e.g., Ref. 2, pp. 30–31) such that assumptions Eqs. (20) and (21) are less restrictive than first appear. This point is amplified in the next section.

The transform of Eq. (20), after multiplication by s and inclusion of Eq. (21), is

$$sg^*(s) = A + \frac{B}{s} + \sum_{\rho=1}^{m} \frac{a_\rho}{1 + b_\rho/s} \tag{22}$$

Values of b_ρ are now taken to be the first m of the M elements of the previously selected sequence $s_\rho: b_\rho = s_\rho, \rho = 1,2, \ldots, m$; $M = m + 2$. The manner of choosing this sequence (which must clearly be done even before $g^*(s_\rho)$ can be explicitly obtained) is still best deferred to the next section where numerical examples are considered. However selected, the M elements s_ρ and associated elements $g^*(s_\rho)$ are substituted consecutively in Eq. (22), resulting in a set of M simultaneous nonhomogeneous linear algebraic equations in a total of M unknown quantities: A, B, and m quantities a_ρ. These may be solved, in principle, by any standard routine. However, it is possible to determine A and a priori B since the limiting forms of $g(t)$ for zero and indefinitely large t are usually known. In any case, it is always possible to determine B beforehand by means of the equation

$$\lim_{s \to 0} s^2 g^*(s) = B \tag{23}$$

which follows immediately from Eq. (22). Further, since the initial temperature is taken to be zero, Eqs. (20) and (21) imply that

$$A = -\sum_{\rho=1}^{m} a_\rho \tag{24}$$

This expression for A may be substituted in Eq. (22) prior to solving for a_ρ.

Illustrative Examples

As a check on the accuracy of the approximations described in the foregoing sections, we first consider the problem of a uniform circular cylinder, initially at zero temperature, subjected to the convection boundary condition (4). For convenience, the parameters $\varkappa$, T_0, and the cylinder radius R are all taken to be unity. Further, $(h/k)R = 2$ and the boundary of the circle is divided into $N = 24$ equal intervals.

Table 1 Temperature variation in cylinder

Time ($\log_{10}t$)	$r = 0$ Analytical	$r = 0$ Numerical	$r = 0.5$ Analytical	$r = 0.5$ Numerical	$r = 1.0$ Analytical	$r = 1.0$ Numerical
−4.0	0	−0.002	0	−0.001	0.028	0.026
−3.6	0	−0.001	0	−0.001	0.037	0.034
−3.2	0	0.001	0	0	0.055	0.052
−2.8	0	0.001	0	0.001	0.085	0.086
−2.4	0	−0.001	0	−0.001	0.131	0.133
−2.0	0	0.001	0	0.001	0.199	0.199
−1.6	0	−0.001	0.005	0.005	0.295	0.295
−1.2	0.007	0.007	0.061	0.061	0.423	0.423
−0.8	0.134	0.135	0.249	0.248	0.584	0.584
−0.4	0.517	0.516	0.591	0.590	0.780	0.780
0	0.896	0.897	0.912	0.912	0.953	0.953
0.4	0.998	0.998	0.998	0.998	0.999	0.999
0.8	1.000	1.000	1.000	1.000	1.000	1.000

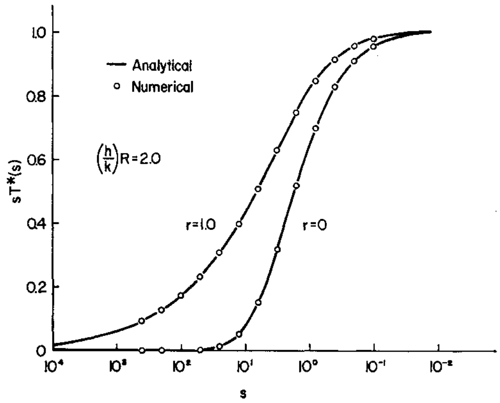

Fig. 1 Transformed temperature in the cylinder.

Next, we choose a sequence of transform parameter values s_ρ to obtain the required sequences of transform data as indicated earlier. Schapery[14] shows that if a plot of $sg^*(s)$ vs log s can be obtained, the significant range of s needed in the inversion scheme, i.e., the first and last terms of the sequence s_ρ, can be chosen by inspection. For example, the variation of $sT^*(s)$ with log s for two locations in the cylinder is shown in Fig. 1. Clearly, the significant range of s referred to is between a minimum of about −0.1 and a maximum of about 100 for one curve, and 0.1 to 1000.0 for the other curve. Outside of these ranges $sT^*(s)$ is essentially constant. Schapery then suggests choosing a sequence with s_1 equal to the minimum significant value such that $s_{\rho+1}/s_\rho = 10.0$ with a sufficient number of terms m such that the significant range of s is covered. With s_ρ so selected, the outlined inversion process may be used to obtain $T(t)$ at a point p as required.

Note that this manner of selecting s_ρ presupposes that a plot of $sg^*(s)$ is available for purposes of selecting the significant range of s. However, it is always possible to choose a sequence s_ρ rather arbitrarily, plot $sg^*(s)$ based on the arbitrary choice, and subsequently choose the sequence necessary for inversion as described by Schapery.

For the cylinder problem and also for the illustrative problems to be subsequently considered, a sequence of $m = M = 14$ terms is chosen beginning with $s_1 = 0.1$ such that $s_{\rho+1}/s_\rho = 2.0$. This choice represents a departure from Schapery's work in that he uses only a ratio $s_{\rho+1}/s_\rho = 10.0$. There appears to be no reason to limit the ratio to 10.0. Indeed, compared to the transient responses obtained with a ratio of 2.0, unsatisfactory responses were obtained using a ratio of 10.0. Other ratios between 10.0 and 2.0 were tried, and, generally, results improved with decreasing ratio.

Using the sequence s_ρ with a ratio of 2.0, transformed temperature data are compared with the analytical transformed solution (Ref. 12, p. 329) in Fig. 1 for two values of the distance r from the center of the cylinder. The accuracy of the transformed solution for $N = 24$ is such that there is a difference from the analytical solution only in the third significant figure. This accuracy increases with increasing N. Numerical data representing the *transient* temperature response based on the ratio $s_{\rho+1}/s_\rho = 2.0$ are compared in Table 1 with the corresponding analytical values (Ref. 2, p. 202) for three distances from the center of the cylinder. Tabuler comparison of transient data is chosen to display the differences which arise again, at most, in the third significant figure; however, it should be noted that the increments in time shown are chosen for brevity. A *continuous* approximate time solution is provided by Eq. (20) such that comparison with analytical data may be made for all t, and indeed, the accuracy shown in Table 1 is typical.

A Gauss-Jordan matrix inversion scheme was used to solve for the coefficients a_ρ in Eq. (22). As Schapery indicates, and as might be expected, accuracy in determining the transient response generally improves with the number of elements

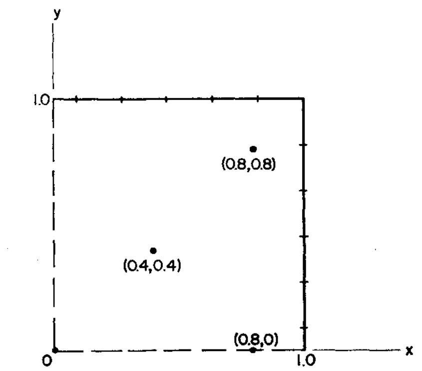

Fig. 2 Quadrant of square region.

m of the sequence. Therefore, by varying the ratio, one can vary the value of m in covering a given significant range of s; whereas, using a fixed ratio, e.g., 10.0 as Schapery does, the value of m is determined unless the range is exceeded. The ratio of 2.0 thus fits the range 0.1 to 1000.0 with a larger number (m = 14) of terms than the mentioned higher ratios.

The problem of a uniform square prism is considered next, the lateral surface of which is subjected to a uniform, constant temperature $f = 1$. The boundary of the square is divided into $N = 44$ equal intervals, each corner lying at a boundary between intervals as shown in Fig. 2. Note that points P_η thus never occupy a corner, as mentioned earlier. Using the same sequence s_ρ as used in the previous problem, numerical data in the form $T(t)$ for four locations in the square are given in Tables 2 and 3. Comparative results were obtained from the analytical solution in Ref. 2, p. 173. We remark, in passing, that a comparison of the quality of the transformed temperature solution, at the same locations p, with the corresponding analytical values[2] was generally as favorable as for the preceding problems.

It is important to note here that having obtained $T(t)$ at p for uniform constant boundary conditions G in the preceding problems, the solution $v(p,t)$ for time dependent boundary conditions $G(t)$ may be given (c.f. Ref. 2, p. 31) directly by

$$v(p,t) = G(0)T(p,t) + \int_0^t T(p,\lambda) \frac{\partial G}{\partial t} (p,t - \lambda)d\lambda \quad (25)$$

with no additional approximation whatever, i.e., for analytically expressed $G(t)$ the integral [Eq. (25)] may be evaluated exactly. In reality then, the outlined procedures yield the solution for arbitrary, uniform, time dependent G. Solutions for nonuniform G, i.e., a function of Q as well as t, may be similarly built up by means of the expression

$$v(p,t) = \int_0^t \frac{\partial T}{\partial t} (p,\lambda,t - \lambda)d\lambda \quad (26)$$

which is the solution for $G(Q,t)$ applied at $t = 0$. However, unlike the case for uniform G, additional approximation would be involved since $T(p,\lambda,t)$ continuous in λ cannot be obtained by the outlined procedures. For such cases it would, perhaps, be well to abandon Schapery's technique in favor of one (Ref. 12,16) not requiring assumptions Eqs. (20) and (21).

Finally, the capability of the solution procedure to handle a boundary condition of mixed type is shown by the solution of a modified version of the previous problem. The modification involves insulating perfectly two opposing sides ($x = \pm 1$) of the prism, while the other two sides are subjected to a uniform, constant unit temperature as before. Thus, temperature is prescribed over part of the boundary, while (0) heat flux is prescribed over the remainder. This problem is equivalent to that of a uniform infinite slab, over both surfaces of which a uniform, constant temperature is prescribed. For this problem the boundary of the square is divided into only $N = 20$ intervals in the same manner as for the previous problem. The time variation of temperature at four locations are presented in Table 4 together with corresponding analytical data from Ref. 2, p. 100. Note that although the same sequence s_ρ is used as before, the data comparison is not as good because of the larger boundary subdivision and associated less accurate transform data. However, even though $N = 20$ represents a quite crude approximation the accuracy is acceptable for most purposes and such accuracy for small N is perhaps surprising despite the relative simplicity and symmetry of the problem. An additional point to be made here is that errors in the transform data obtained as described lead to comparable errors in the transient solution. This is not always the case for different transform inversion schemes (c.f. Ref. 11, 15).

Concluding Remarks

All numerical data in the transform space were obtained by taking advantage of symmetry. Specifically, elements of the **A** and **B** matrices in Eq. (14) were obtained only for points P_η occupying positions in the first quadrant of the domain boundaries C. Thus only $N/4$ of Eqs. (14) were actually formed and solved on the computer. In general, if no symmetry is present, dealing with the full system of N equations is still practicable in light of the small computer times (IBM 360-50) involved. For example, to solve the $N/4$ system of algebraic equations for fourteen values of the transform parameter for

Table 2 Temperature variation in square prism

	$x = 0$, $y = 0$		$x = 0.4$, $y = 0.4$	
Time ($\log_{10}t$)	Analytical	Numerical	Analytical	Numerical
−2.0	0	−0.001	0	0
−1.6	0	0.001	0.015	0.015
−1.2	0.019	0.019	0.174	0.173
−0.8	0.280	0.279	0.509	0.508
−0.4	0.773	0.773	0.851	0.851
0	0.988	0.988	0.992	0.992
0.4	1.000	1.000	1.000	1.000

Table 3 Temperature variation in square prism

	$x = 0.8$, $y = 0$		$x = 0.8$, $y = 0.8$	
Time ($\log_{10}t$)	Analytical	Numerical	Analytical	Numerical
−3.0	0	−0.003	0	−0.008
−2.6	0.005	0.003	0.010	0.007
−2.2	0.075	0.074	0.144	0.142
−1.8	0.261	0.257	0.454	0.444
−1.4	0.479	0.473	0.728	0.723
−1.0	0.672	0.669	0.881	0.877
−0.6	0.854	0.852	0.955	0.954
−0.2	0.978	0.977	0.993	0.992
0.2	1.000	1.000	1.000	0.999

Table 4 Temperature variation in square prism with two insulated sides

	$y = 0$			$y = 0.8$		
	Analytical	Numerical		Analytical	Numerical	
Time ($\log_{10}t$)	(all x)	$x = 0$	$x = 1.0$	(all x)	$x = 0$	$x = 1.0$
−3.0	0	0.001	0.001	0	−0.002	0.001
−2.4	0	−0.001	−0.001	0.025	0.032	0.016
−1.8	0	0	0	0.261	0.260	0.227
−1.2	0.010	0.010	0.010	0.573	0.562	0.569
−0.6	0.317	0.318	0.321	0.787	0.783	0.794
0	0.892	0.895	0.897	0.967	0.966	0.970
0.6	1.000	1.000	1.000	1.000	1.000	1.000

the cylinder problem with $N = 24$ required only three minutes. Also, to obtain a_p for the inversion process and 26 values of $T(t)$ at one point p required, typically, less than $\frac{1}{2}$ min for fourteen terms.

With regard to the conditioning of the key algebraic Eqs. (14), we remark that they are similar in character but essentially simpler than Eqs. (10) in Ref. 17 [also Eqs. (35) in Ref. 18] where the conditioning effect of increases in N, presence of corners, and changes in boundary interval spacing are more fully discussed. It suffices to remark here that the mere presence of corners using the outlined techniques causes no difficulty with all data (see Tables 3 and 4) obtained near and on the boundary itself. Rounding of corners, as is necessary for example, in Eq. (19) where a similar singular integral technique is used, is not required here. Significant increases in N, on the other hand, lead first to impractical demands on computer capacity that are unnecessary because of the negligible accuracy increases that occur long before ill-conditioning is experienced. In regard to spacing changes, ill-conditioning can result if interval spacing changes significantly. However, this can be circumvented by scaling of certain of the Eqs. (19) before attempting to solve them as described in Ref. 17.

Finally, in regard to the popular finite difference and related methods of approximate solution, the following remarks are in order. In the present analysis, a sequence of transformed temperature data need be obtained only at field points where the transient solution is actually desired. This is accomplished by an interval-node§ network only on the boundary of the domain, where irregularity of shape is totally unimportant. Further, as was shown, N need not be large. In effect, then, a two-dimensional problem involves a one-dimensional spatial approximation with an associated number N (assuming no symmetry) of algebraic equations. This is in contrast to a network of the order N^2 required for a finite-difference procedure where a) N frequently must be large for accuracy,¶ b) the form of the difference equations must be altered depending on the boundary conditions, and c) special attention to preserve accuracy is required for irregular boundaries, i.e., the boundary intersects the network strings. In the present scheme, the first and fundamental approximation is that of piecewise constant T^* and $\partial T^*/\partial n$ on C such that following the solution of a comparatively small number of algebraic equations, the interior T^* is generated only where desired with no additional approximation in an insignificant amount of time compared to that required to solve the algebraic equations. Indeed, all that is required to obtain T^* for a chosen p and s_p is to sum [Eq. (15)] $2N$ products of known quantities. The only restriction, for obvious reasons, is that T^* is not sought in D closer to C than one nodal spacing. This is not a serious limitation, however, since T^* on C is previously known from the solution of Eq. (14).

With both the present and finite-difference procedures the respective algebraic equations must be solved sequentially, i.e., for values of s_p in one case and for increments in time in the other. However, for the class of problems involving boundary conditions as described, a relatively small number of s_p is required to yield, via Schapery's technique, an accurate expression for the temperature, continuous in time, such that solutions for time dependent boundary conditions are given directly by Eq. (25).

The illustrative problems of the last section were chosen, primarily, to check the feasibility of the proposed ideas and to check the accuracy of the approximations against the associated analytical solutions. Our intent is to demonstrate that the present scheme is indeed a workable one with a potential for obtaining accurate numerical transform data and, hence, by Schapery's or some other inversion process, yield a powerful and effective method of solution for a variety of problems. Its greatest potential is, perhaps, for three-dimensional problems, where because of the effective reduction in dimension involved, the computational problems, compared to finite difference schemes, would be less than enormous. Such problems and ones involving anisotropy of heat conduction are currently being investigated.

§ Interval points $Q_{\xi\pm1/2}$, nodal points P_η.

¶ Notwithstanding the frequent sparsity of the matrices.

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